cation savings factor decreases with the increasing number of subfilters. The limit of this approach is when all \((N/L)\) subfilters per subsection are used, in which case the filter is matched to the LFM waveform and \(2 \cdot N \cdot K\) multiplications are required.

VI. Conclusions

Nonrecursive digital filters can be synthesized by several techniques, as described by various authors. Furthermore, the complexity of the filter may depend on the synthesis technique chosen. This paper has presented a technique for designing some types of finite-duration impulse-response digital filters that require many fewer multiplications than other known techniques. Any desired finite-duration impulse response can be synthesized with this approach. However, the savings in multiplications depend on the impulse responses desired. For some well-known radar applications, such as pulse compression, the multiplication savings factor can be very large. The examples discussed in this paper indicate that the savings in the required multiplication rate can be factors of 100 or more.

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References


Randomly Sampled Digital Filters

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Abstract

The concept of a randomly sampled digital filter is introduced and formulated in this paper. An equivalent problem, which is of utmost importance in digital filtering, is also considered, namely, the problem of digital filtering with faulty samplers. Error analysis is carried out in two different approaches: 1) frequency response method, and 2) worst-case analysis. With the confirmation of experimental verification, very simple formulas are derived that are applicable under different realistic assumptions, and are most appealing to practical design purposes.

I. What is a Randomly Sampled Digital Filter?

The term digital filter has been well understood [1] to be a system that processes an input \(\{x_k\}_{k=0}^\infty\) and transforms it to an output \(\{w_k\}_{k=0}^\infty\) according to the following linear difference equation:

\[
w_n = \sum_{i=0}^{M} b_i x_{n-i} - \sum_{i=1}^{N} a_i w_{n-i}, \quad \forall n.
\]

There are two interpretations of the operation of this system: 
a) the two sequences \(\{x_k\}\) and \(\{w_k\}\) may be considered as just any sequences of numbers without any implication of time ordering, in which case the process of digital filtering is just the execution of the algorithm (1) [see Fig. I(A)]; and 
b) these sequences \(\{x_k\}\) and \(\{w_k\}\) may be considered as samples, as in sampled-data control theory, of continuous signals \(x(\cdot)\) and \(w(\cdot)\), in which case, a time ordering is implied; and this time ordering is usually uniform, with samples spaced at \(T\) 's apart [see Fig. I(B)]. With this latter interpretation, the idea of a randomly sampled digital filter can be introduced as follows.

Consider then a continuous input signal \(x(\cdot)\), randomly sampled by a sampler which closes once in every sampling period of duration \(T\). (See Fig. 2.) A sequence of numbers \(\{s_k\}_{k=0}^\infty\) is thus evolved, where \(s_k = x(t_k)\), for all \(k, t_k\) being the sampling instant in the \(k\)th sampling period. This sequence of numbers can be processed directly algorithmically as in Fig. I(A). Otherwise, in order to introduce a time ordering, the sequence \(\{s_k\}\) can be stored (in a computer) and

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periodically retrieved sequentially, say with a period of $T$, giving the time series $\{x(mT)\}_{m=0}^{\infty}$ (for an arbitrary $m_0$). In this case, $x(k+m_0T) = x_0$, $\forall k$. And, $\{x(mT)\}$ can be processed as in Fig. 2(B), as if $\{x(mT)\}$ had been uniformly sampled from a perturbed continuous input signal $x(\cdot)$.

This device of a randomly sampled digital filter is by no means a contrived one. Actually, it is not difficult to conceive that it is only natural to introduce this idea of random sampling when, say, digital filters are time shared. And of course, for this time sharing to operate rationally, each channel (input to the filter) must share its use of the computer (i.e., the filter) once in every $T$-s interval, a feature imposed on the randomly sampled digital filter in the previous paragraph. Also, such randomness does take place, say, in radar tracking of a rapidly moving evader, also in which case a sample of the evader's position is recorded once every sweep cycle (as long as the object remains in screen). In fact, the ideas of random sampling in feedback control systems have been introduced earlier and found to be realistic models of certain physiological systems. (See [2], [3].) This reveals then, that random sampling should be regarded as an integral part of the digital filtering technology.

**II. An Equivalent Problem: Previous Works and Present Investigation**

Instead of viewing this randomly sampled digital filter as a new device, the following equivalent problem is recognized: the randomness of the sampling process may be considered as an unintentional process as opposed to an intentional process, i.e., as a natural phenomenon versus a contrived feature. In other words, when a faulty sampler is present, the designer would like to find out how much this faultiness affects the system performance. Error analysis of this faulty system is then naturally the first step towards the study of this system.

It is only timely here to point out that Balakrishnan [4] performed an intensive analysis of the problem of “time jittering in sampling” back in 1962. In that paper, the spectrum of the sampled signal (i.e., the randomly sampled signal) is computed for the cases of deterministic and of stationary stochastic signals. The analysis of the latter case is relatively straightforward, since the derived (i.e., sampled) signal is again stationary. As to the former case of deterministic input, it was noted that the sampled signal is not stationary. To overcome this difficulty of nonstationarity, an average spectrum was defined, which then brought the analysis of this case into parallelism with that of the other case. As digital filtering is fundamentally a linear process, the output of the filter is thus well defined through a spectrum shaping of the sampled signal. Thus, Balakrishnan’s study essentially spelled out the whole error analysis of the randomly sampled digital filter, save the complex nature of his formulas.

The purpose of this paper is to use a different approach, essentially an engineer’s practical approach, to tackle the problem of error analysis of such filters in the case of deterministic input. And, instead of defining the average spectrum and propagating this spectrum through the filter, the following two approaches are taken.

1) **Frequency Response Method**: A modified transfer function is derived and its expected value computed, which is then plotted against frequency. Also the average steady state noise-to-signal ratio is computed and displayed in frequency-plots.

2) **Worst-Case Analysis**: Bounds on the expected values of the error and its energy are expressed in terms of the maximum input “velocity” and “acceleration.”

Formulas derived on these two analyses are found to be very simple and most easily computed; and the validity of these are checked by experimental verifications. Furthermore, the derived formulas depend on some practically accessible quantities such as the input’s harmonic contents, or the bounds on the input’s derivatives. And thus, the present analyses are seen to be most attractive to practical design purposes.
III. Formulation

Referring back to Fig. 2(A), the deviation of the random sampling instants from the uniform instants is defined as follows:

\[ t_n = n + Z_n, \quad \forall n, \]  

where \( T \), the sampling period, has been normalized to unity, and \( Z_n \) is a random variable taking values in \([ -\Delta, \Delta] \), with \( \Delta = aT/2, \ 0 \leq a < 1 \). \( Z_n \) is assumed to be evenly distributed, with the result that all the odd moments vanish. Typical distributions are 1) rectangular, 2) triangular, and 3) trapezoidal with parameter \( p, \ 0 \leq p \leq 1 \). These distributions are shown in Fig. 3. Their first four even moments are tabulated in Table I. Formally, the system assumptions are as follows.

A. System Assumptions

Assumption 1: \( x(t) \) is a deterministic signal, assumed to be infinitely differentiable.

Assumption 2: \( \{ h_n \}_{n=-\infty}^{\infty} \) is the impulse response of the digital filter, assumed to be strictly stable.

Assumption 3: \( T \), the sampling period, is normalized to unity.

Assumption 4: \( Z_n \) is a random variable evenly distributed in \([ -\Delta, \Delta] \), with

\[ \Delta = aT/2, \quad 0 \leq a < 1, \quad \forall n. \]

\( Z_n \) is independent of \( Z_m, \quad \forall n \neq m \).

Assumption 5: \( \{ Z_n \}_{n=-\infty}^{\infty} \) is a stationary stochastic process, with \( \sigma^2 \) and \( \eta^4 \) denoting \( E(Z_n^2) \) and \( E(Z_n^4) \), respectively.

B. Basic Analysis

By Assumption 1, \( x(t_n) = \hat{x}_n \) is expanded as

\[ \hat{x}_n = x(n + Z_n) = x_n + \hat{x}_n Z_n + \frac{1}{2} \hat{x}_n Z_n^2 + \cdots. \]  

Define the input error \( \{ \varepsilon_n \} \)

\[ \varepsilon_n = \hat{x}_n - x_n \]

\[ = \hat{x}_n Z_n + \frac{1}{2} \hat{x}_n Z_n^2 + \cdots. \]

Since the digital filter is a linear system, the output error \( \{ e_n \} \) is

\[ e_n = y_n - w_n \]

\[ = \sum_{i=0}^{n} h_{n-i} \varepsilon_i \]

where \( \{ w_n \} \) and \( \{ y_n \} \) denote the outputs of the uniformly sampled system and the randomly sampled system respectively. (See Figs. 1 and 2.) With the assumption on \( \{ Z_n \} \), the following expressions are readily obtained.

1 By the terminology used in [4], \( \{ Z_n \} \) is said to be "pure white."
IV. Error Analysis

In this section, the input is assumed to be a cosinusoid, i.e.,
\[ x(t) = \cos \omega t \] (7)
with \( 0 \leq \omega \leq N_f \). Here \( N_f \) denotes the Nyquist frequency, which equals \( \frac{\pi}{T} = \pi \text{ rad/s} \). This range \([0, N_f]\) is the largest frequency band in which the digital filter, while uniformly sampled, will operate meaningfully, and thus indicates the plausibly largest operating range of frequencies for the randomly sampled digital filter. In fact, the following will show that, depending on the distribution of \( \theta \), only a portion of \([0, N_f]\) is operational, in the sense of tolerable noise-to-signal ratio (NSR). Denote \( \beta = \omega / N_f \).

Noting that \( \xi(t) = (-\omega \sin \omega t, -\omega \cos \omega t, \ldots, (\omega^2 \xi) \) and (c) can be rewritten as
\[ E(\xi_n) = (-\frac{1}{2} \omega^2 \eta^2) \cos n\omega + \cdots \] (9a)
\[ E(\xi_n^2) = (\omega^2 \eta^2) \sin^2 n\omega + (\omega^4 \eta^4)(\frac{1}{4} \cos^2 n\omega - \frac{1}{3} \sin^2 n\omega) + \cdots. \] (9b)

In order to see whether these infinite series can be truncated at the terms indicated, it is necessary to study the convergence of the terms \( \omega^p E(Z_n^p), \omega^p E(Z_n^p), \omega^p E(Z_n^p), \ldots \). Taking the example of rectangular distribution, \( \omega^p E(Z_n^p) \) can be rewritten
\[ \omega^p E(Z_n^p) = (\beta N_f)^p (\frac{1}{4} \Delta^p) \]
\[ = (\beta \pi / T)^2 \cdot \frac{1}{2} (\alpha T/2)^2 \]
\[ = \frac{\pi^2}{12} (\beta \alpha)^2. \] (10)

Note the independence of \( T \) and the dependence on the product parameter \( \beta \). A list of these quantities for the three distributions is tabulated in Table II. Reasonably fast convergence is easily seen to be obtainable when: 1) \( \beta \alpha \leq 0.5 \) for the rectangular distribution, 2) \( \beta \alpha \leq 0.75 \) for the triangular distribution, and 3) intermediate ranges for \( \beta \alpha \) for the trapezoidal distribution depending on the parameter \( p \). It is further revealed by a comment at the end of this section that useful operation of the randomly sampled digital filter usually requires that the frequency ranges be restricted such that these bounds on \( \beta \alpha \) are satisfied. Thus, approximation of (9) by the indicated truncated series are valid in the operational frequency ranges. Rewriting then,
\[ E(\xi_n) = (-\frac{1}{2} \omega^2 \eta^2) \cos n\omega \] (11a)
\[ E(\xi_n^2) = (\omega^2 \eta^2) \sin^2 n\omega + (\omega^4 \eta^4)(\frac{1}{4} \cos^2 n\omega - \frac{1}{3} \sin^2 n\omega) \]
\[ = (\frac{1}{4} \omega^2 \eta^2 - \frac{1}{3} \omega^4 \eta^4) \]
\[ + (-\frac{1}{2} \omega^2 \eta^2 + \frac{1}{2} \omega^4 \eta^4) \cos n(2\omega). \] (11b)

And, the output to error \( e_n \) has its first two moments
\[
\begin{array}{c|c|c|c}
\hline
\text{Rectangular} & \text{Triangular} & \text{Trapezoidal with} \\
\hline
\omega^p E(Z_n^p) & (\frac{\pi^2}{12} (\beta \alpha)^2) & (\frac{\pi^2}{24} (\beta \alpha)^2) & (\frac{5}{72} \pi^2 (\beta \alpha)^2) \\
\hline
\omega^p E(Z_n^p) & (\frac{\pi^4}{80} (\beta \alpha)^4) & (\frac{\pi^4}{240} (\beta \alpha)^4) & (19 \pi^4 (\beta \alpha)^4) \\
\hline
\omega^p E(Z_n^p) & (\frac{\pi^6}{448} (\beta \alpha)^6) & (\frac{\pi^6}{1792} (\beta \alpha)^6) & (85 \pi^6 (\beta \alpha)^6) \\
\hline
\omega^p E(Z_n^p) & (\frac{\pi^8}{2304} (\beta \alpha)^8) & (\frac{\pi^8}{11520} (\beta \alpha)^8) & (1223 \pi^8 (\beta \alpha)^8) \\
\hline
\end{array}
\]

The steady-state expressions are recognized to be
\[ E(\xi_n) = (-\frac{1}{2} \omega^2 \eta^2) \sum_{i=0}^{n} h_{n-i} \cos i\omega \] (12a)
\[ E(\xi_n^2) = E^2(\xi_n) + \sum_{i=0}^{n} h_{n-i}^2 \cos i(2\omega) \]
\[ + (-\frac{1}{2} \omega^2 \eta^2 + \frac{1}{2} \omega^4 \eta^4) \sum_{i=0}^{n} h_{n-i}^2 \cos i(\omega) \]
\[ + \frac{1}{4} \omega^2 \eta^2 - \frac{1}{3} \omega^4 \eta^4 \]
\[ \cdot \sum_{i=0}^{n} h_{n-i}^2 \cos i(2\omega). \] (12b)

where \( \sum_{i=0}^{n} h_{n-i}^2 = h_{n}^2 \), and \( \hat{H}(\omega) = \sum_{i=0}^{n} h_{n-i} z^{-i} \). From (13a), a "modified function" \( H_m(z) \) can be defined describing the input-output relation of the randomly sampled digital filter excited by a cosine input, as if the input were still uniformly sampled but the transfer function of the filter were modified. And, the first moment of \( H_m(z) \) is given by
\[ E(H_m(z)) = (1 - \frac{1}{2} \omega^2 \eta^2) \hat{H}(\omega), \quad z = e^{i\omega}. \] (14)

Notice that only the expected amplitude response is distorted, while its expected phase response remains unchanged. As to the second moment, i.e., the energy contained in the error signal, a meaningful measure is to find the NSR:
\[ \text{NSR}(\omega) \triangleq 10 \log_{10} \frac{P_e(\omega)}{P_m(\omega)} \text{ dB} \] (15)
at steady state, where $P_u(\omega)$, $P_v(\omega)$ denote the average expected power contained in $\{e_n\}$ and $\{w_n\}$ at their steady states, corresponding to an input of frequency $\omega$, and with the average taken over a sufficiently large number of samples. For example,

$$
(w_n)_\omega = H(z \mid z = e^{j\omega}) \cos \omega
$$

with $\phi \triangleq H(z \mid z = e^{j\omega})$, and

$$
(w_n)_\omega^2 = |H(z \mid z = e^{j\omega})|^2 \cos^2 (\omega + \phi).
$$

Averaging $(w_n)_\omega$ over a “cycle,”

$$
P_u(\omega) = |H(z \mid z = e^{j\omega})|^2 \frac{1}{2}. \tag{18}
$$

In the same way, averaging $E(e_n^2)_\omega$ over the same cycle,

$$
P_v(\omega) = (-\frac{1}{2})^2 P_v(\omega)
$$

Thus, by (15), the NSR is given by

$$
NSR(\omega) = 10 \log_{10} \left\{ 1 + \frac{2|h|^2 \left( \frac{3}{2} \omega^2 \phi^2 - \omega^2 \phi \omega^2 - \frac{1}{2} \omega^4 \phi^2 \right)}{|H(e^{j\omega})|^2} \right\} \text{dB}. \tag{20}
$$

Simplification of (20) is possible when $\omega^2 \phi^2 \ll |h|^2 \omega^2 / |H(e^{j\omega})|^2$ and $\omega^4 \phi^2 \ll \omega^2 \phi^2$. This leads to

$$
NSR(\omega) = 10 \log_{10} \left\{ |h|^2 \left( \frac{3}{2} \omega^2 \phi^2 - \frac{1}{2} \omega^4 \phi^2 \right) / |H(e^{j\omega})|^2 \right\} \text{dB}. \tag{21}
$$

This simplification can easily be seen to offer good approximation in the operational frequency range (see comment following the example, concerning the truncation of infinite series) and when the filter does not have excessive high gains. As a matter of fact (21) is used instead of (20) in the following example.

Example: Take the filter having the following transfer function $H(z) = 0.1 / (1 - 0.9 z^{-1})$. By (14) and (21), $|H(e^{j\omega})|^2$ and $E[H_n(e^{j\omega})]^2$ and NSR are plotted for: 1) rectangular distribution, 2) triangular distribution, and 3) trapezoidal with $p = \frac{1}{2}$ distribution of $Z_n$.

Comment

With regard to the operational frequency range and the truncation of the infinite series in (6), here is the following observation. Taking the case of rectangular distribution and

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3 When the sampling rate is large compared to the input frequency, $|\cos n \omega T|$, behaves like $\cos \omega (\cdot)$. This cycle, then, can be taken over $2\pi / \omega$. Otherwise, a larger interval must be chosen accordingly.
\(\alpha = 1\) in the above example (refer to curve \(i\) in Fig. 4), the NSR becomes intolerable \((-5\, \text{dB})\) for \(\beta = \omega / \Omega > 0.3\).
(The degree of tolerance depends on the application of the filter, say for transmission of voices or transmission of pictures, in which cases the former can tolerate much heavier distortion than the latter. Typical figures are \(-10\, \text{dB}\) for the former and \(-30\, \text{dB}\) for the latter.) That means that successful operation of the randomly sampled digital filter is not available beyond \(\beta > 0.3\) in the present illustration. (This is certainly plausible, in regard of the extreme wildness of the random sampling when \(\alpha = 1\).) By this observation, and recalling an earlier assertion that the truncated expressions \((11)\) and \((12)\) are valid approximations of \((6)\) in the range \(0 \leq \alpha \leq \beta \leq 0.5\) in the present case, it is then seen that these truncations are indeed legal in the operational frequency range \(0 \leq \beta \leq 0.3\). Similar agreements for other cases thus justify completely the truncations of the infinite series in \((6)\).

V. Error Analysis (II): Worst-Case Analysis

In this section, upper bounds on the expected error and its variance are obtained on the assumption of knowledge of the bounds on the input velocity and acceleration. It is to be noted that these input quantities are easily accessible and practically controllable. In addition, it is assumed here that \(x(t)\) is smooth enough and \(Z_{\alpha}\) small enough such that \(x_{\alpha}^{(2k)} E(Z_{\alpha}^{(2k)})\) goes to zero with \(k\) reasonably fast, where \(x_{\alpha}^{(2k)}\) denotes the \((2k)\)th derivative of \(x(\cdot)\). This assumption would be easily satisfied in common practice when \(T\) is small (of the order of milliseconds). Formally, here are the extra assumptions.

Assumption 6: \(|x(t)| \leq c_1, |x(t)| \leq c_2, \forall t\).

Assumption 7: \(x_{\alpha}^{(2k)} E(Z_{\alpha}^{(2k)}) \equiv 0\) reasonably fast, \(\forall n\).

Then, \((4)\) can be truncated and written as

\[E(e_{\alpha}) = E^2(e_{\alpha}) + \left(\sum_{i=0}^{n} h_{\alpha-\epsilon} e_{\alpha}\right)^2 + \frac{1}{2} \left(\sum_{i=0}^{n} h_{\alpha-\epsilon}^2 e_{\alpha}^2\right) \eta^4 - \nu^4.\]  

Thus,

\[E(e_{\alpha}) = \frac{1}{2} \nu^2, \quad E^2(e_{\alpha}) = \frac{1}{2} \nu^2 + \frac{1}{2} \eta^4 \nu^4, \quad \text{var}(e_{\alpha}) = \frac{1}{2} \nu^2 + \frac{1}{2} \eta^4 \nu^4.\]  

Note that \(\eta^4 - \nu^4 \geq 0\) because

\[\text{var}(Z_{\alpha}^4) = E(Z_{\alpha}^4) - E^2(Z_{\alpha}^4) \geq 0.\]

And, the first two moments of the output error \(e_{\alpha} = \sum_{i=0}^{n} h_{\alpha-\epsilon} e_{\alpha}\) are given by the following expressions:

\[E(e_{\alpha}) = \frac{1}{2} \left(\sum_{i=0}^{n} h_{\alpha-\epsilon} e_{\alpha}\right)^2, \quad \text{var}(e_{\alpha}) = \frac{1}{2} \left(\sum_{i=0}^{n} h_{\alpha-\epsilon} e_{\alpha}^2\right) \eta^4 - \nu^4.\]  

These expressions are bounded by

\[|E(e_{\alpha})| \leq \frac{1}{2} c_1 \nu^2, \quad E^2(e_{\alpha}) \leq c_2 \nu^2 + \frac{1}{2} c_2 \eta^4 \nu^4, \quad \text{var}(e_{\alpha}) \leq \frac{1}{2} \eta^4 \nu^4 + \nu^4.\]  

VII. Conclusion

The idea of randomly sampled digital filters has been introduced and formulated in this paper. Its importance and natural occurrence are pointed out. Also, it is unveiled that an equivalent problem is the very important aspect of digital filtering with faulty samplers. It is recognized that the latter problem has been very rigorously analyzed by Balakrishnan in the context of time jittering in sampling, who arrived at exact analytic expressions of the spectral densities of the randomly sampled signals in the case of deterministic input as well as stochastic input. However, the present paper aims at and successfully arrives at relatively much simpler expressions for the error analysis of such filters when excited by a smooth deterministic signal. The derived expressions are approximations, which are proved to be valid under various realistic assumptions. As a matter of fact, two approaches are proposed here: 1) frequency response method, and 2) worst-case analysis. These are indeed familiar tools to the design engineers. The first approach arrives at frequency plots of the expected modified transfer function and the output noise-to-signal ratio as a function of input frequency. The second approach assumes a knowledge of the bounds on the input velocity and acceleration—also practically accessible quantities—and arrives at upper bounds on the possible expected error and its variance. Evidence of experimental verification also confirms the theory. All these then lead to the conclusion that the present analyses are most suitable and appealing to practical engineering design purposes.

References